JHEP HYPER VERSION

Received: *August 13, 2001,* Accepted: *November 14, 2001*

Quantum determinism for free vector bosons in 3 dimensions

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Abstract: We apply 't Hooft's deterministic quantum mechanics approach to free vector bosons in three dimensions and check Lorentz invariance. This approach does not work for the conformal group, for free bosons in two dimensions. This presents a technical difficulty for constructing a "deterministic string theory".

Keywords: Conformal Field Models in String Theory, Gauge Symmetry, Global Symmetries, Models of Quantum Gravity.

JHEP11(2001)028

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1. Introduction

In a series of papers, 't Hooft has advocated a deterministic approach to quantum mechanics. For a review, see [1]. We were motivated by his suggestion to use his procedure for string theory (see [1, bottom of page 6]). In [2], 't Hooft introduces a new kind of (non-local) gauge symmetry. It removes the amplitude of the field momentum modes, keeping only their phases. As explained in [2], the classical free boson field with this gauge symmetry is equivalent to a quantum free boson. Furthermore, except for the zero modes, the spectrum of each momentum mode can be truncated to a finite number of states.

In this note, we point out a technical difficulty in carrying out this program for string theory: the "global gauge symmetry" is not compatible with the conformal group. It is therefore not obvious how to follow through with the general philosophy. We should quickly point out that there might be clever ways around this problem, and that 't Hooft's deterministic approach remains very interesting. Our argument is very simple: a conformal transformation involves an arbitrary function $\epsilon(z)$ and it is not possible to reabsorb the effect of this transformation with a transformed kernel that is independent of the world-sheet space coordinate of the scalar field.

In section 2, we review the Lorentz invariance of the scalar; in section 3, we show the Lorentz covariance of the free vector boson in three dimensions; in section 4, we show that the approach works for dilatations for the free boson in two dimensions but not for other conformal transformations.

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2. Lorentz invariance

The reader looking for new results should skip this section.

2.1 Boosts

As a warm-up, we review 't Hooft's check that a real scalar field remains invariant under Lorentz transformations if the kernels transform in a certain way [3]. The purpose of presenting these details here is that the calculations of the later sections are very similar, so we will be able to skip some details later. Consider a real scalar free field $\phi(x,t)$ of mass μ in $D+1$ dimensions, satisfying $\ddot{\phi} = (\triangle - \mu^2)\phi$. Under 't Hooft's gauge transformation, ϕ and $\dot{\phi}$ become:

$$\begin{aligned}
\phi^{(g)}(x,t) &= \int d^D y \; K_1(y)\phi(x+y,t) + K_2(y)\dot{\phi}(x+y,t)\,, \\
\dot{\phi}^{(g)}(x,t) &= \int d^D y \; K_1(y)\dot{\phi}(x+y,t) + K_2(y)\left(\triangle_x - \mu^2\right)\phi(x+y,t)\,,
\end{aligned} \tag{2.1}$$

using the equations of motion to replace $\ddot{\phi}$ by $(\triangle - \mu^2)\phi$. Are the gauge transformed $\phi^{(g)}$ and $\dot{\phi}^{(g)}$ still Lorentz invariant? Under an infinitesimal boost in the x_1 direction,

$$\begin{aligned}
\delta\phi(x,t) &= x_1\dot{\phi}(x,t) + t\partial_{x_1}\phi(x,t)\,, \\
\delta\dot{\phi}(x,t) &= x_1\left(\triangle - \mu^2\right)\phi(x,t) + t\partial_{x_1}\dot{\phi}(x,t) + \partial_{x_1}\phi(x,t)\,.
\end{aligned} \tag{2.2}$$

We can set $t=0$ at this stage: if ϕ and $\dot{\phi}$ transform properly at $t=0$, they will remain so at later times by the equations of motion. But we will keep t explicitly to illustrate that such terms follow through straightforwardly. If we assume that the kernels transform as follows under this boost:

$$\begin{aligned}
\delta K_1(y) &= -y_1\left(\triangle_y - \mu^2\right)K_2(y) - \partial_{y_1}K_2(y)\,, \\
\delta K_2(y) &= -y_1 K_1(y)\,,
\end{aligned} \tag{2.3}$$

then $\phi^{(g)}$ and $\dot{\phi}^{(g)}$ transform correctly:

$$\begin{aligned}
\delta\phi^{(g)}(x,t) &= x_1\dot{\phi}^{(g)}(x,t) + t\partial_{x_1}\phi^{(g)}(x,t)\,, \\
\delta\dot{\phi}^{(g)}(x,t) &= x_1\left(\triangle - \mu^2\right)\phi^{(g)}(x,t) + t\partial_{x_1}\dot{\phi}^{(g)}(x,t) + \partial_{x_1}\phi^{(g)}(x,t)\,.
\end{aligned} \tag{2.4}$$

To check that, just substitute eqs. (2.2) and (2.3) into:

$$\begin{aligned}
\delta\phi^{(g)}(x,t) &= \int \; K_1\delta\phi + (\delta K_1)\phi + K_2\delta\dot{\phi} + (\delta K_2)\dot{\phi}\,, \\
\delta\dot{\phi}^{(g)}(x,t) &= \int \; K_1\delta\dot{\phi} + (\delta K_1)\dot{\phi} + K_2\delta\left(\left(\triangle - \mu^2\right)\phi\right) + (\delta K_2)\left(\triangle - \mu^2\right)\phi\,.
\end{aligned} \tag{2.5}$$

Note that δ acts on $\triangle$ in the above equation:

$$\delta\left(\left(\triangle - \mu^2\right)\phi\right) = \left(\triangle - \mu^2\right)\delta\phi + 2\partial_{x_1}\dot{\phi}\,. \tag{2.6}$$

For the reader's convenience, we give the details of the computation; substituting:

$$\begin{aligned}\delta\phi^{(g)}(x,t) = \int d^D y\ K_1(y)\Big((x_1+y_1)\dot\phi(x+y) + t\partial_{x_1}\phi(x+y)\Big) - \\ - \left(y_1\left(\triangle_y-\mu^2\right)K_2(y) + \partial_{y_1}K_2(y)\right)\phi(x+y) + \\ + K_2(y)\Big((x_1+y_1)\left(\triangle_x-\mu^2\right)\phi(x+y) + t\partial_{x_1}\dot\phi(x+y) + \\ + \partial_{x_1}\phi(x+y)\Big) - y_1K_1(y)\dot\phi(x+y)\,.\end{aligned}$$

The terms containing K_1 cancel in part to leave $x_1\int K_1\dot\phi + t\partial_{x_1}\int K_1\phi$. For the terms containing K_2, after integrations by parts and discarding the surface terms, so that all the derivatives are with respect to the variable x, we get $x_1(\triangle_x-\mu^2)\int K_2\phi + t\partial_{x_1}\int K_2\dot\phi$. They combine to give:

$$\begin{aligned}\delta\phi^{(g)}(x,t) &= x_1\int\Big(K_1\dot\phi + K_2\left(\triangle_x-\mu^2\right)\phi\Big) + t\partial_{x_1}\int\Big(K_1\phi+K_2\dot\phi\Big) \\ &= x_1\dot\phi^{(g)} + t\partial_{x_1}\phi^{(g)}\,.\end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned}\delta\dot\phi^{(g)}(x,t) &= \int d^D y\ K_1(y)\Big((x_1+y_1)\left(\triangle_x-\mu^2\right)\phi(x+y) + \\ &\qquad + t\partial_{x_1}\dot\phi(x+y) + \partial_{x_1}\phi(x+y)\Big) - \\ &\quad - \left(y_1\left(\triangle_y-\mu^2\right)K_2(y) + \partial_{y_1}K_2(y)\right)\dot\phi(x+y) + \\ &\quad + K_2(y)\left(\triangle_{x+y}-\mu^2\right)\Big((x_1+y_1)\dot\phi(x+y) + t\partial_{x_1}\phi(x+y))\Big) + \\ &\quad + 2K_2(y)\partial_{x_1}\dot\phi - y_1K_1(y)\left(\triangle_x-\mu^2\right)\phi(x+y) \\ &= \left(x_1\left(\triangle-\mu^2\right)+\partial_{x_1}\right)\times \\ &\quad \times\int\Big(K_1\phi+K_2\dot\phi\Big) + t\partial_{x_1}\int\Big(K_1\dot\phi + K_2\left(\triangle-\mu^2\right)\phi\Big) \\ &= x_1\left(\triangle-\mu^2\right)\phi^{(g)}(x,t) + t\partial_{x_1}\dot\phi^{(g)}(x,t) + \partial_{x_1}\phi^{(g)}(x,t)\,.\end{aligned} \tag{2.8}$$

2.2 Rotations and translations

Under translations, the kernels are invariant: $\delta K_{1,2}=0$. Under rotations, the kernels transform as scalars. That is, for a rotation in the $1-2$ plane,

$$\begin{aligned}\delta\phi(x,t) &= (x_1\partial_{x_2} - x_2\partial_{x_1})\phi(x,t)\,, \\ \delta K_{1,2}(y) &= (y_1\partial_{y_2} - y_2\partial_{y_1})K_{1,2}(y)\,.\end{aligned} \tag{2.9}$$

3. The free vector boson in $2+1$ dimensions

In [3], it is shown how the quantum deterministic approach works for a free vector boson in $3+1$ dimensions. The free vector boson in $2+1$ dimensions is simpler, but not entirely trivial. The key is understanding what equation 5.3 in [3] becomes.Lorentz invariance works for the vector in $2+1$ dimensions because it can be dualized to a scalar. We do not know how to generalize these examples to more than four dimensions: because the little group is larger, it is not clear how to write a decomposition such as equation 5.3 in [3].

We now show how the vector in $2+1$ dimensions works in some detail. We write the duality equation $F_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho\phi$ in momentum space. With the momentum in the direction $k_2 = k$, we have $k_1 = 0$ and the frequency $\omega = |k|$. In the gauge $A_0 = 0$, there is the constraint $\partial_i A^i = 0$, which is $k_1 A_1 + k_2 A_2 = 0$, and so $A_2(k) = 0$ too.

$$\begin{aligned} F_{01} &= \partial_0 A_1 = \partial_2\phi\,, \\ F_{02} &= \partial_0 A_2 = -\partial_1\phi\,, \\ F_{12} &= -\partial_2 A_1 = -\partial_0\phi\,. \end{aligned} \tag{3.1}$$

In momentum space:

$$\begin{aligned} \dot{A}_1 &= ik\phi\,, \\ \dot{A}_2 &= 0\,, \\ ikA_1 &= \dot{\phi}\,. \end{aligned} \tag{3.2}$$

Thus A_2 stays zero and A_1 satisfies its equation of motion: $\ddot{A}_1 = ik\dot{\phi} = -k^2 A_1$. We now want ϕ, $\dot{\phi}$ to transform in the usual way. In momentum space, this is [2]:

$$\begin{aligned} \phi^{(g)} &= K_1\phi + iK_2\dot{\phi}\,, \\ \dot{\phi}^{(g)} &= K_1\dot{\phi} - iK_2 k^2\phi\,. \end{aligned} \tag{3.3}$$

And therefore, by eq. (3.2), A_1 transforms as:

$$\begin{aligned} A_1^{(g)} &= K_1 A_1 + iK_2\dot{A}_1\,, \\ \dot{A}_1^{(g)} &= K_1\dot{A}_1 - ik^2 K_2 A_1\,. \end{aligned} \tag{3.4}$$

The Lorentz invariance of $\phi^{(g)}$ will carry over to $A^{(g)}$. Let's see this explicitly. Returning to coordinate space:

$$\begin{aligned} A_1^{(g)}(x,t) &= \int d^2y\; K_1(y)A_1(x+y,t) + K_2(y)\dot{A}_1(x+y,t)\,, \\ \dot{A}_1^{(g)}(x,t) &= \int d^2y\; K_1(y)\dot{A}_1(x+y,t) + K_2(y)\triangle_x A_1(x+y,t)\,. \end{aligned} \tag{3.5}$$

Since $\partial_i A_i = 0$, we can rewrite it as:

$$\begin{aligned} A_i^{(g)}(x,t) &= \int d^2y\; K_1(y)A_i(x+y,t) + L_1(y)\dot{A}_i(x+y,t)\,, \\ \dot{A}_i^{(g)}(x,t) &= \int d^2y\; K_1(y)\dot{A}_i(x+y,t) + L_1(y)\triangle A_i(x+y,t)\,, \end{aligned} \tag{3.6}$$

where we relabeled $L_1 = K_2$ to conform to the notation of [3].

The rest of the proof of Lorentz invariance can be read off from [3] by setting $K_2 = L_2 = 0$ and dropping one coordinate.

$$\begin{aligned} \delta A_1 &= x_2 \dot{A}_1 + t\partial_2 A_1 + \partial_1 \Lambda\,, \\ \delta A_2 &= x_2 \dot{A}_2 + t\partial_2 A_2 + \partial_2 \Lambda\,, \\ \delta A_0 &= A_2 + \partial_0 \Lambda\,, \end{aligned} \tag{3.7}$$

so $\Lambda = -\triangle^{-1}\dot{A}_2$ and $\partial_0\Lambda = -A_2$. At $t = 0$:

$$\begin{aligned} \delta A_i &= x_2 \dot{A}_i - \triangle^{-1}\partial_i \dot{A}_2\,, \\ \delta \dot{A}_i &= x_2 \triangle A_i + \partial_2 A_i - \partial_i A_2 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \delta K_1(y) &= -y_2 \triangle_y L_1(y) - \partial_{y_2} L_1(y)\,, \\ \delta L_1(y) &= -y_2 K_1(y)\,. \end{aligned} \tag{3.9}$$

4. Conformal invariance

We now consider whether 't Hooft's procedure is applicable to conformal invariance in two dimensions. We will find out that it is not directly applicable.

We can work with world-sheet coordinates $z = \sigma + i\tau$ and $\bar{z}$, say on the cylinder. Under a conformal transformation,

$$z \longrightarrow z + \epsilon(z)\,, \tag{4.1}$$

where $\epsilon(z)$ is a holomorphic function. The free world-sheet periodic boson $X(z,\bar{z})$ satisfies $\partial\bar{\partial}X(z,\bar{z}) = 0$, so it splits into left-movers $X_L(z)$ and right-movers $X_R(\bar{z})$; $X(z,\bar{z}) = X_L(z) + X_R(\bar{z})$; $\dot{X} = i(\partial X_L - \bar{\partial}X_R)$. It is sufficient to look at the left-movers for our argument. Under (4.1),

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\Phi &= \left(h\partial\epsilon + \epsilon\partial + \bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\right)\Phi(z,\bar{z})\,, \\ \delta X_L(z) &= \epsilon(z)\partial X_L(z)\,, \\ \delta\partial X_L(z) &= \epsilon(z)\partial^2 X_L(z) + (\partial\epsilon(z))X_L(z)\,. \end{aligned} \tag{4.2}$$

Under 't Hooft's gauge transformation,

$$\begin{aligned} X_L^{(g)}(z) &= \int dy\ K_1(y)X(z+y) + iK_2(y)\partial_z X(z+y)\,, \\ i\partial_z X_L^{(g)}(z) &= \int dy\ K_1(y)i\partial_z X(z+y) - K_2(y)\partial_z^2 X(z+y)\,, \end{aligned} \tag{4.3}$$

with y a real, one-dimensional variable. The transformed $X_L^{(g)}$ and $\partial X_L^{(g)}$ are still holomorphic. However, they do not transform properly under conformal transformations. For example, take $\epsilon(z) = \epsilon_n z^n$, for an integer $n > 2$. Then

$$\begin{aligned}\delta X_L^{(g)} = &\int dy\ \delta K_1 X + i\delta K_2 \partial X + K_1 \epsilon_n (z+y)^n \partial X + \\ &+ iK_2 \epsilon_n \left((z+y)^n \partial^2 X + n(z+y)^{n-1} \partial X\right).\end{aligned} \tag{4.4}$$

Expanding $(z+y)^n$ using the binomial theorem, we see that the z^n term is what we want to keep for the Lorentz transformation of X. However, the subleading terms $z^{n-k} y^k$ are z-dependent terms. Using integrations by parts, one can hope at best to remove two subleading powers of z from the second-derivative ∂_z^2. The rest of the z dependence remains, and therefore the variation of the kernels $K_{1,2}$ would have to depend on z, which is not allowed.

4.1 Special conformal transformations

For the global conformal group, the procedure also does not work for special conformal transformations ($n = 2$). Although the unwanted terms can be cancelled by variations of the kernels, these variations do not have the right symmetries under $y \to -y$. We use a prime to denote a σ derivative. There are two kinds of special conformal transformations. The first type, at $\tau = 0$:

$$\begin{aligned}\delta X &= \sigma^2 X', \\ \delta \dot{X} &= \sigma^2 \dot{X}' + 2\sigma \dot{X}.\end{aligned} \tag{4.5}$$

In $\delta X^{(g)}$, collect the terms without time derivatives:

$$\delta X^{(g)} = \sigma^2 \partial_\sigma \int K_1 X + \left(\int \delta K_1 X - \partial_y \left(K_1 \left(\underline{y^2} + 2y\sigma\right)\right) X\right) + \cdots. \tag{4.6}$$

The underlined term, $\partial_y (K_1 y^2) X$ is odd under $y \to -y$ and has the wrong symmetry property for a variation of K_1. The second type:

$$\begin{aligned}\delta X &= \sigma^2 \dot{X}, \\ \delta \dot{X} &= \sigma^2 X'' + 2\sigma X'.\end{aligned} \tag{4.7}$$

In $\delta X^{(g)}$, collect the terms with time derivatives:

$$\delta X^{(g)} = \sigma^2 \int K_1 \dot{X} + \left(\int \delta K_2 \dot{X} + \left(K_1 \left(\underline{y^2} + 2y\sigma\right)\right) \dot{X}\right) + \cdots. \tag{4.8}$$

The underlined term, $K_1 y^2 \dot{X}$ is even under $y \to -y$ and has the wrong symmetry property for a variation of K_2.

4.2 Dilatations

The procedure does finally work for dilatations (and of course boosts). Under a dilatation at $\tau = 0$,

$$\begin{aligned} \delta X &= \sigma X' \,, \\ \delta \dot{X} &= \sigma \dot{X}' + \dot{X} \,. \end{aligned} \tag{4.9}$$

The variations of the kernels are:

$$\begin{aligned} \delta K_1(y) &= (y K_1(y))' \,, \\ \delta K_2(y) &= y K_2'(y) \,. \end{aligned} \tag{4.10}$$

Acknowledgments

This work was supported by NSF grant PHY-0071512.

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